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A Sampling Theorem for Duration-Limited Functions with Error Estimates

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In contrast to the classical Shannon sampling theorem, signal functions are considered which are not band-limited but duration-limited. It is shown that these functions can be approximately represented by a discrete set of samples. The error is estimated that arises when only a finite number of samples is selected.

1. INTRODUCTION

The well-known sampling theorem set up by Whittaker (1915), Kotel'nikov (1933), and Shannon (1940/1949), plays a basic role in communication, control, and data processing. It states that every real-valued deterministic signal $f(t)$ that is band-limited to $[-\pi W, \pi W]$ can be reconstituted by

$$\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \qquad (t \in \mathbb{R}). \tag{1.1}$$

Since by Fourier transform theory band-limited signals must necessarily extend for an infinite time and, conversely, a signal which only exists for a certain time must have a spectrum (Fourier transform) which contains frequencies up to infinity, Wunsch (1963) and Kioustelidis (1969) have considered another model for a sampling theorem, one which is just as practical, namely to reconstruct duration- (or time-) limited functions from samples. Whereas in the Wunsch paper no proof of the corresponding result is supplied, arguments in the proof of the Kioustelidis paper are not complete; see also the review of the latter paper.[1] Furthermore, neither author mentioned hypotheses upon the signal function under which their sampling theorem would hold. Nevertheless, the Kioustelidis paper inspired the present authors to establish a sampling theorem for continuous duration-limited functions whose spectrum

$$f^{\wedge}(v) = (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} f(t)\, e^{-ivt}\, dt \qquad (v \in \mathbb{R}) \tag{1.2}$$

[1] Doetsch, G. (1970), *Math. Reviews 40*, 4716.

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belongs to $L^1(\mathbb{R}) = \{g; \int_{-\infty}^{\infty} |g(t)|\, dt < \infty\}$, the class of functions that are absolutely integrable (in sense of Lebesgue). This is the main aim of this paper.

Several authors, including Cybakov and Jakovlev (1959), Jagerman (1966), Yao and Thomas (1966), and Piper (1975) have treated the so-called truncation error that arises when the signal function is reproduced by only a finite number of terms from its Shannon sampling series expansion (1.1) rather than by the full infinite series. Therefore an error estimate, a counterpart of this truncation error, is presented for the alternative model of the sampling theorem. One of these estimates is given in terms of functions of bounded variation (see Butzer and Oberdörster, 1975), the other in terms of signals satisfying additional smoothness conditions.

The latter estimates are dealt with in Section 4. For these results several lemmas are needed, considered in Section 2. Lemma 1 will also be used to prove the Shannon sampling theorem under simple and precise conditions upon the signal function which do not involve an expansion of the transform $f^\wedge$ into its Fourier series (see also Balakrishnan, 1968), the classical procedure employed in most books on transmission of information or signal theory (e.g., Hölzler and Holzwarth (1975), Rosie (1973), Panter (1965), Franks (1969)). The Shannon and the alternative model of the sampling theorem are treated in Section 3. The proofs use only real variable methods, namely those of Fourier analysis and approximation theory.

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2. Some Preliminary Lemmas

In order to establish the sampling theorems, namely Theorems 1 and 2, the following Fourier series expansion of the 2π-periodic extension $e(x)$ of the function e^{itWx}, $x \in [-\pi, \pi)$, $t \in \mathbb{R}$, $W > 0$, is important.

Lemma 1. *Let $t \in \mathbb{R}$, $W > 0$ with $tW \notin \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and*

$$\begin{aligned} e(x) &= e^{itWx}, & x &\in [-\pi, \pi), \\ &= e^{itW(x+2\pi j)}, & x &\in [-(2j+1)\pi, -(2j-1)\pi), \end{aligned} \tag{2.1}$$

where $j \in \mathbb{Z}$. Then one has for all $x \in \mathbb{R}$, $x \neq (2j+1)\pi$, $j \in \mathbb{Z}$,

$$e(x) = \sum_{k=-\infty}^{\infty} e^{ikx} \frac{\sin \pi(Wt - k)}{\pi(Wt - k)}, \tag{2.2}$$

the sum in (2.2) converging boundedly on $\mathbb{R}$.

Proof. The Fourier coefficients of e are equal to

$$
\begin{aligned}
e_c^{\wedge}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{itWx}e^{-ikx}\,dx = \frac{1}{2\pi W}\int_{-\pi W}^{\pi W} e^{i(t-k/W)x}\,dx \\
&= \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \qquad (k \in \mathbb{Z}).
\end{aligned} \tag{2.3}
$$

Since e is continuous on $(-\pi, \pi)$ and, since

$$e(x) = \int_{-\pi}^{x} itWe^{itWu}\,du + e^{-i\pi tW} \qquad (x \in [-\pi, \pi)),$$

also absolutely continuous on $(-\pi, \pi)$, it follows by a well-known result on Fourier series (e.g., Hardy and Rogosinski, 1956, p. 32) that the sequence of partial sums

$$(S_n e)(x) = \sum_{k=-n}^{n} e^{ikx}\,\frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \tag{2.4}$$

of the Fourier series of e is uniformly bounded on $\mathbb{R}$, and converges pointwise to

$$
\begin{cases}
e(x), & x \in \bigcup_{j\in Z} (-(2j+1)\,\pi, -(2j-1)\,\pi), \\
\cos \pi Wt, & \text{otherwise.}
\end{cases} \tag{2.5}
$$

This proves Lemma 1. (Note that (2.2) as well as the other series expansions in this paper are trivial if $tW \in \mathbb{Z}$.)

Let $C(\mathbb{R})$ be the class of those functions that are continuous on $\mathbb{R}$ with $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$. For the next result we need to define the Lipschitz class $\mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$; it is the set of functions $f \in C(\mathbb{R})$ for which there exists a constant $L > 0$ (the so-called Lipschitz constant) such that

$$\|f(\cdot + h) - f(\cdot)\| \leqslant L\,|h|^{\alpha} \qquad (h \in \mathbb{R}). \tag{2.6}$$

Defining the (first) modulus of continuity of $f \in C(\mathbb{R})$ by

$$\omega_1(\delta; f) = \sup_{|h|\leqslant\delta} \|f(\cdot + h) - f(\cdot)\|,$$

then definition (2.6) is equivalent to

$$\mathrm{Lip}\,\alpha = \{f \in C(\mathbb{R});\ \omega_1(\delta; f) \leqslant L\delta^{\alpha},\ \delta > 0\}.$$

LEMMA 2. *Let $f \in C(\mathbb{R})$ such that $f(t) = 0$ for all $|t| > T$ for fixed $T > 0$, and $f^{\wedge} \in L^1(\mathbb{R})$. Then $f^{\wedge} \in \mathrm{Lip}\,1$ with*

$$\|f^{\wedge}(\cdot + h) - f^{\wedge}(\cdot)\| \leqslant \frac{1}{(2\pi)^{1/2}}\, T^2\, \|f\|\, |h| \qquad (h \in \mathbb{R}).$$

Proof. Since f is uniformly continuous on $[-T, T]$, $\|f\|$ is finite. By hypothesis one therefore has that

$$\begin{aligned}(2\pi)^{1/2}\,|f^\wedge(v+h)-f^\wedge(v)| &= \left|\int_{-T}^{T} f(t)[e^{-i(v+h)t}-e^{-ivt}]\,dt\right| \\ &= \left|\int_{-T}^{T} f(t)\,e^{-ivt}e^{-iht/2}(-2i)\sin\frac{ht}{2}\,dt\right| \\ &\leqslant 4\,\|f\|\int_0^T \left|\sin\frac{ht}{2}\right| dt \leqslant T^2\,\|f\|\,|h|,\end{aligned}$$

which completes the proof since the bound is independent of $v \in \mathbb{R}$.

For the sampling theorem for duration-limited functions we also need

LEMMA 3. *Under the assumptions of Lemma* 2 *one has for* $W > 0$

$$\left|\int_{|v|>\pi W} f^\wedge(v)\,e^{ivk/W}\,dv\right| \leqslant \frac{2\,\|f\|}{(2\pi)^{1/2}}\,(\pi WT)^2\,\frac{1}{|k|} \qquad (|k| > TW). \tag{2.7}$$

Proof. Now $f^\wedge \in \mathrm{Lip}\,1$ on account of Lemma 2. So for each $v \in \mathbb{R}$ there exists a function $\epsilon(v, h)$ and a constant $L = T^2\,\|f\|/(2\pi)^{1/2} > 0$ such that

$$\begin{gathered} f^\wedge(v+h) = f^\wedge(v) + \epsilon(v,h) \\ \|\epsilon(\cdot,h)\| \leqslant L\,|h| \qquad (h \in \mathbb{R}). \end{gathered} \tag{2.8}$$

Since both f and $f^\wedge$ belong to $L^1(\mathbb{R})$, $f(t)$ can be represented as a Fourier integral

$$f(t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f^\wedge(v)\,e^{ivt}\,dv \tag{2.9}$$

valid for all $t \in \mathbb{R}$ as $f \in C(\mathbb{R})$. Since $f(k/W) = 0$ for all $|k| > TW$, one therefore has for these k that

$$\int_{|v|>\pi W} f^\wedge(v)\,e^{ivk/W}\,dv = -\int_{-\pi W}^{\pi W} f^\wedge(v)\,e^{ivk/W}\,dv. \tag{2.10}$$

Subdividing the interval $[-\pi W, \pi W]$ into $|k|$ equal parts and introducing the notation $w(j, k) = -\pi W + 2\pi Wj/|k|$, one has with (2.8) that

$$\begin{aligned}\int_{-\pi W}^{\pi W} f^\wedge(v)\,e^{ivk/W}\,dv &= \sum_{j=0}^{|k|-1}\int_{w(j,k)}^{w(j+1,k)} f^\wedge(v)\,e^{ivk/W}\,dv \\ &= \sum_{j=0}^{|k|-1}\int_{w(j,k)}^{w(j+1,k)} \epsilon(w(j,k), v - w(j,k))\,e^{ivk/W}\,dv.\end{aligned}$$

To establish this one uses the identity

$$\int_{w(j,k)}^{w(j+1,k)} e^{ivk/W}\, dv = 0$$

which is valid since $e^{ivk/W}$ is $2\pi W/|k|$-periodic. So on account of (2.10) and (2.8)

$$\left| \int_{|v|>\pi W} f^\wedge(v)\, e^{ivk/W}\, dv \right| \leqslant \sum_{j=0}^{|k|-1} L \int_0^{2\pi W/|k|} h\, dh \leqslant 2(\pi W)^2 L \frac{1}{|k|},$$

as was to be shown.

3. Sampling Theorem for Duration—Limited Functions

Before proving this theorem let us establish for the sake of completeness the sampling theorem for band-limited functions with the radian bandwidth $\pi W/2$.

Theorem 1. *Let $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $f^\wedge(v) = 0$ for all $|v| > \pi W$, $W > 0$. Then*

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \qquad (t \in \mathbb{R}). \tag{3.1}$$

Proof. Since f is low-pass band-limited, one has in (2.9) that

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\pi W}^{\pi W} f^\wedge(v)\, e^{ivt}\, dv \tag{3.2}$$

for all $t \in \mathbb{R}$ since $f^\wedge \in L^1(\mathbb{R})$ by hypothesis. Substituting (2.2) for $x = v/W$ into (3.2) yields

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\pi W}^{\pi W} f^\wedge(v) \sum_{k=-\infty}^{\infty} e^{ivk/W} \frac{\sin \pi(Wt-k)}{\pi(Wt-k)}\, dv \qquad (t \in \mathbb{R}).$$

By Lemma 1 one can apply Lebesgue's theorem on dominated convergence to interchange the order of integration and summation to give assertion (3.1) on account of (3.2).

A similar proof has been given by Balakrishnan (1968) for signal functions which are quadratically integrable. Other modes of proof can be found, e.g., in Churkin *et al.* (1966) using function theory or in Brown (1968) and Jerri (1973) using the Parseval and Hölder inequalities for $L^p(\mathbb{R})$-functions, $1 \leqslant p < \infty$. For various generalizations see also Nathan (1973) and Bar-David (1974).

As mentioned in the Introduction, signal functions cannot be simultaneously band-limited and duration-limited. So let us consider a sampling theorem for functions having the latter property, i.e., which vanish outside a specified time interval $[-T, T]$. If a function $f(t)$ is to be determined by its sampled values $f(k/W)$, $k \in \mathbb{Z}$, for a given $W > 0$, then $2N + 1$ such values must be evaluated, $N = N(T, W)$ being the largest integer less than TW. Then the function f is the limit for $W \to \infty$ of the corresponding sum. Indeed,

THEOREM 2. *Let* $T, W > 0$ *and* $f \in C(\mathbb{R})$ *such that* $f(t) = 0$ *for all* $|t| > T$ *and* $f^\wedge \in L^1(\mathbb{R})$. *Then*

$$\lim_{W\to\infty} \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi(Wt - k)}{\pi(Wt - k)} = f(t) \qquad (t \in \mathbb{R}). \tag{3.3}$$

Proof. Let g be the $2\pi W$-periodic extension of

$$f^\wedge(v) = \frac{1}{(2\pi)^{1/2}} \int_{-T}^{T} f(t)\, e^{-ivt}\, dt \qquad (v \in [-\pi W, \pi W)).$$

Then the Fourier coefficients of g have the form ($k \in \mathbb{Z}$)

$$g_c^\wedge(k) = \frac{1}{2\pi W} \int_{-\pi W}^{\pi W} f^\wedge(v)\, e^{-ivk/W}\, dv \tag{3.4}$$

$$= \frac{(2\pi)^{1/2}}{2\pi W} \left\{ f\left(-\frac{k}{W}\right) - \frac{1}{(2\pi)^{1/2}} \int_{|v|>\pi W} f^\wedge(v)\, e^{-ivk/W}\, dv \right\} \tag{3.5}$$

by (2.9). Since $f^\wedge \in \mathrm{Lip}\, 1$ by Lemma 2, there is $M > 0$ with $|g(v + h) - g(v)| \leqslant M |h|$, $|h| < \pi(W - W')$, uniformly in $v \in [-\pi W', \pi W']$, any fixed $W' \in (0, W)$. As a corollary of Dini's test (see Hardy and Rogosinski, 1956, p. 41), the Fourier series of g is uniformly convergent on $[-\pi W', \pi W']$ to $f^\wedge(v)$, and one has by (3.5)

$$\begin{aligned} f^\wedge(u) = &\sum_{k=-N}^{N} \frac{(2\pi)^{1/2}}{2\pi W} f\left(\frac{k}{W}\right) e^{-iuk/W} \\ &- \sum_{k=-\infty}^{\infty} \frac{1}{2\pi W} \left(\int_{|v|>\pi W} f^\wedge(v)\, e^{-ivk/W}\, dv \right) e^{iuk/W} \end{aligned} \tag{3.6}$$

uniformly for $u \in [-\pi W', \pi W']$. On account of the uniform convergence one may integrate term-by-term to give (as in (2.3))

$$\begin{aligned} &\frac{1}{(2\pi)^{1/2}} \int_{-\pi W'}^{\pi W'} f^\wedge(u)\, e^{iut}\, du \\ &= \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi W'(t - k/W)}{\pi(Wt - k)} \\ &- \sum_{k=-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \left(\int_{|v|>\pi W} f^\wedge(v)\, e^{ivk/W}\, dv \right) \frac{\sin \pi W'(t - k/W)}{\pi(Wt - k)}. \end{aligned}$$

It therefore follows by (2.9) that for each $W' \in (0, W)$ the error ($t \in \mathbb{R}$)

$$\begin{aligned} R_W(t) &:= f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \\ &= \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) \frac{\sin \pi W'(t-k/W) - \sin \pi(Wt-k)}{\pi(Wt-k)} \\ &+ \frac{1}{(2\pi)^{1/2}} \int_{|v|>\pi W'} f^{\wedge}(v)\, e^{ivt}\, dv \\ &- \sum_{k=-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \left(\int_{|v|>\pi W'} f^{\wedge}(v)\, e^{ivk/W}\, dv \right) \frac{\sin \pi W'(t-k/W)}{\pi(Wt-k)}. \end{aligned} \tag{3.7}$$

The first two terms on the right-hand side of (3.7) are continuous functions of W'. The third term is a series of continuous functions which is uniformly convergent with respect to W' on account of Lemma 3 and since

$$\left| \frac{\sin \pi W'(t-k/W)}{\pi(Wt-k)} \right| \leqslant \frac{2}{\pi} \frac{1}{|k|} \qquad (|k| > 2tW).$$

Letting therefore $W' \to W$ in (3.7) one has

$$\begin{aligned} R_W(t) &= \frac{1}{(2\pi)^{1/2}} \int_{|v|>\pi W} f^{\wedge}(v)\, e^{ivt}\, dv \\ &- \sum_{k=-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \left(\int_{|v|>\pi W} f^{\wedge}(v)\, e^{ivk/W}\, dv \right) \frac{\sin \pi(Wt-k)}{\pi(Wt-k)}. \end{aligned} \tag{3.8}$$

Since $f^{\wedge} \in L^1(\mathbb{R})$ and because of Lemma 1 one can again apply Lebesgue's theorem on dominated convergence to (3.8) to yield

$$\begin{aligned} R_W(t) &= \frac{1}{(2\pi)^{1/2}} \int_{|v|>\pi W} f^{\wedge}(v) \left\{ e^{ivt} - \sum_{k=-\infty}^{\infty} e^{ivk/W} \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \right\} dv \\ &= \frac{2}{(2\pi)^{1/2}} \int_{\pi W}^{\infty} \operatorname{Re}\left[f^{\wedge}(v) \left(e^{ivt} - \sum_{k=-\infty}^{\infty} e^{ivk/W} \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \right) \right] dv. \end{aligned} \tag{3.9}$$

This yields that

$$|R_W(t)| \leqslant \frac{2}{(2\pi)^{1/2}} \int_{\pi W}^{\infty} |f^{\wedge}(v)| \left| e^{ivt} - \sum_{k=-\infty}^{\infty} e^{ivk/W} \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} \right| dv$$

and so by (2.5)

$$|R_W(t)| \leqslant \frac{4}{(2\pi)^{1/2}} \int_{\pi W}^{\infty} |f^{\wedge}(v)|\, dv \qquad (t \in \mathbb{R}). \tag{3.10}$$

Since $f^{\wedge} \in L^1(\mathbb{R})$, this gives assertion (3.3).

Let us mention a paper[2] by Brown (1967) which showed: If $f(t) = (1/2\pi)\int_{-\infty}^{\infty} f^\wedge(v)\, e^{ivt}\, dv$ with $f^\wedge \in L^1(\mathbb{R})$, then

$$\left| f(t) - \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2W}\right) \frac{\sin \pi(2Wt-k)}{\pi(2Wt-k)} \right| \leqslant \frac{1}{\pi} \int_{|v|>2\pi W} |f^\wedge(v)|\, dv.$$

Theorem 2 could be obtained as a special case of this result. Basic for our proof (see Lemma 2) as well as for that of Theorem 3, however, is the assumption that the signal be of finite length, a fact which is natural in the applications.

4. Error Estimates

Let us now determine the rate of convergence in (3.3). At first we consider signal functions f, their derivative f' being of bounded variation, that means, simply stated f' can be represented by a curve of finite length over $[-T, T]$ or

$$[\mathrm{Var}\, f']_{-T}^{T} := \int_{-T}^{T} |\, df'(t)| < \infty$$

($\int_{-T}^{T} g(t)\, df(t)$ being the Lebesgue–Stieltjes integral of g with respect to f over $[-T, T]$). By integrating by parts twice we have

$$\begin{aligned}(2\pi)^{1/2} f^\wedge(v) &= f(t)\frac{e^{-ivt}}{-iv}\Big|_{-T}^{T} + \int_{-T}^{T} f'(t)\frac{e^{-ivt}}{iv}\, dt \\ &= \frac{1}{v^2}\left(f'(T+0)\, e^{-ivT} - f'(-T-0)\, e^{ivT}\right) - \frac{1}{v^2}\int_{-T}^{T} e^{-ivt}\, df'(t)\end{aligned}$$

(where $f'(T+0) = \lim_{t\to T, t<T} f'(t)$, $f'(-T-0) = \lim_{t\to -T, t>-T} f'(t)$), which leads to

$$|f^\wedge(v)| \leqslant \frac{1}{(2\pi)^{1/2}} \frac{1}{|v|^2} \left(|f'(T+0)| + |f'(-T-0)| + [\mathrm{Var}\, f']_{-T}^{T}\right).$$

With (3.10) one therefore has

$$|R_W(t)| \leqslant (2/\pi^2)\left(|f'(T+0)| + |f'(-T-0)| + [\mathrm{Var}\, f']_{-T}^{T}\right)(1/W).$$

Better rates of convergence can be obtained provided that higher derivatives belong to a Lipschitz class. For this purpose it is useful to introduce the rth modulus of continuity of $f \in C(\mathbb{R})$, namely

$$\omega_r(\delta; f) = \sup_{|h|\leqslant\delta} \|\Delta_h^r f(\cdot)\| \qquad (r \in \mathbb{N} = \{1, 2, \ldots\}), \tag{4.1}$$

[2] The authors would indeed like to thank the referee for pointing out this paper to them.

the rth (right) difference being defined by

$$\Delta_h^r f(t) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(t+kh). \tag{4.2}$$

It is known (cf. Butzer and Nessel, 1971 p. 140) that if $f^{(r)} \in C(\mathbb{R})$, then

$$\omega_{r+j}(\delta; f) \leqslant \delta^r \omega_j(\delta; f^{(r)}) \qquad (r, j \in \mathbb{N}). \tag{4.3}$$

For the error that arises when f is determined by its sampled values at the rate of $1/W$ seconds apart one has

THEOREM 3. *In addition to the hypotheses of Theorem 2 let the* rth *derivative* $f^{(r)}$ *belong to* Lip α *with constant* $L^{(r)}$, $r \in \mathbb{N}$, $0 < \alpha \leqslant 1$. *Then*

$$|R_W(t)| \leqslant \frac{2(T+1)L^{(r)}}{2^r(r+\alpha-1)} \frac{1}{W^{r+\alpha-1}} \qquad (W > (r+1)/2). \tag{4.4}$$

In particular, if $f^{(r+1)} \in C(\mathbb{R})$, *then*

$$|R_W(t)| \leqslant \frac{2(T+1)}{r2^r} \| f^{(r+1)} \| \frac{1}{W^r} \qquad (W > (r+1)/2). \tag{4.5}$$

Proof. Noting that $e^{i\pi(2j+1)} = -1$, $e^{i\pi(2j)} = 1$ for $j \in \{0, 1, 2, ...\}$, one has

$$\begin{aligned} f^\wedge(v) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(t)\, e^{-iv(t-2\pi j/v)}\, dt \\ &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(t + 2\pi j/v)\, e^{-ivt}\, dt, \\ -f^\wedge(v) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(t + (2j+1)\pi/v)\, e^{-ivt}\, dt. \end{aligned}$$

Since $f(t) = 0$ for $|t| > T$, it follows that for $v \geqslant \pi W$

$$\begin{aligned} &\frac{1}{(2\pi)^{1/2}} \int_{-T-(r+1)/W}^{T} \sum_{k=0}^{r+1} (-1)^k \binom{r+1}{k} f(t + k\pi/v)\, e^{-ivt}\, dt \\ &\quad = \sum_{k=0}^{r+1} \binom{r+1}{k} f^\wedge(v) = 2^{r+1} f^\wedge(v). \end{aligned}$$

This yields the inequality

$$|f^\wedge(v)| \leqslant 2^{-r-1} \frac{2T + (r+1)/W}{(2\pi)^{1/2}} \omega_{r+1}(\pi/v; f) \qquad (v \geqslant \pi W).$$

Since $f^{(r)} \in$ Lip α, i.e., $\omega_1(\delta; f^{(r)}) \leqslant L^{(r)}\delta^\alpha$, $\delta > 0$, it follows by (4.3) that

$$|f^\wedge(v)| \leqslant 2^{-r-1} \frac{2T + (r+1)/W}{(2\pi)^{1/2}} \frac{L^{(r)}\pi^{r+\alpha}}{v^{r+\alpha}} \qquad (v \geqslant \pi W).$$

Substituting this inequality into the estimate (3.10) delivers

$$\begin{aligned} |R_W(t)| &\leqslant 2^{-r}(2T + (r+1)/W)L^{(r)} \int_{\pi W}^{\infty} v^{-r-\alpha}\,dv \\ &\leqslant \frac{2(T+1)L^{(r)}}{2^r(r+\alpha-1)} \frac{1}{W^{r+\alpha-1}} \qquad (W > (r+1)/2). \end{aligned}$$

In particular, if $f^{(r+1)} \in C(\mathbb{R})$, then $f^{(r)} \in \mathrm{Lip}\ 1$ with the constant $\|f^{(r+1)}\|$, yielding (4.5).

The proceeding Theorems 2 and 3 tell us that the restriction to signals with finite band width but infinite duration in the classical sampling theorem is not necessary. The result that a signal can be approximately reconstructed by a finite number of samples remains valid without the assumption of band-limitedness. Furthermore, if one wants to transmit a signal of finite duration with the sampling principle allowing a maximal error less than a given ϵ, it suffices to take sample values every $2(r\epsilon/(2(T+1)\|f^{(r+1)}\|))^{1/2}$ second. This follows by (4.5) for signal functions with continuous $(r+1)$th derivative. In particular, if $r = 1$, it suffices to take samples every $\epsilon/((T+1)\|f''\|)$ second.

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References

Balakrishnan, A. V. (1968), "Communication Theory," McGraw–Hill, New York.

Bar-David, I. (1974), An implicit sampling theorem for bounded bandlimited functions, *Inform. Contr.* **24**, 36–44.

Brown, J. L., Jr. (1967), On the error in reconstructing a nonbandlimited function by means of the bandpass sampling theorem, *J. Math. Anal. Appl.* **18**, 75–84.

Brown, J. L., Jr. (1968), Sampling theorem for finite energy signals, IEEE Trans. Information Theory **IT-14**, 818–819.

Butzer, P. L., and Nessel, R. J. (1971), "Fourier Analysis and Approximation Vol. I: One-dimensional Theory," Academic Press, New York; Birkhäuser, Basel.

Butzer, P. L., and Oberdörster, W. (1975), Linear functionals defined on various spaces of contiñuous functions on $\mathbb{R}$, *J. Approximation Theory* **4**, 451–469.

Churkin, J. I., Jakovlev, C. P., and Wunsch, G. (1966), "Theorie und Anwendung der Signalabtastung," VEB Verlag Technik, Berlin.

Cybakov, B. S., and Jakovlev, V. P. (1959), On the accuracy of restoring a function with a finite number of terms of a Kotel'nikov series, *Radio Engrg. Electron. Phys.* **4**, 274–275.

Franks, L. E. (1969), "Signal Theory," Prentice–Hall, Englewood Cliffs, N.J.

HARDY, G. H., AND ROGOSINSKI, W. W. (1956), "Fourier Series," 3rd ed., University Press, Cambridge.

HÖLZER, E., AND HOLZWARTH, H. (1975), "Pulstechnik," Springer, Berlin/Heidelberg.

JAGERMAN, D. (1966), Bounds for truncation error of the sampling expansion, *SIAM J. Appl. Math.* **14**, 714–723.

JERRI, A. J. (1973), Sampling theorem for not necessarily finite energy signals, *Int. J. Systems Sci.* **4**, 255–260.

KIOUSTELIDIS, J. B. (1969), Fehlerbetrachtungen beim Abtasttheorem, *Arch. Elek. Übertr.* **23**, 629–630.

KOTEL'NIKOV, V. A. (1933), On the carrying capacity of the "ether" and wire in telecommunications (in Russian), Vsesoinzn. energ. Kom. Materialy K Pervomu Vsesoiusnomu c'ezdu po voprosu rekonstr. dela sviazi i razv. slabotochn. promyshl., *Izd. Red. Upr. Sviazi RKKA.*

NATHAN, A. (1973), On sampling a function and its derivatives, *Inform. Contr.* **22**, 172–182.

PANTER, P. F. (1965), "Modulation, Noise, and Spectral Analysis," McGraw–Hill, New York.

PIPER, H. S., JR. (1975), Bounds for truncation error in sampling expansions of finite energy band-limited signals, *IEEE Trans. Information Theory* **IT-21**, 482–485.

ROSIE, A. M. (1973), "Information and Communication Theory," Van Nostrand, Reinhold, New York.

SHANNON, C. E. (1949), Communication in the presence of noise, *Proc. IRE* **37**, 10–21.

WHITTAKER, E. T. (1915), On the functions which are represented by the expansion of the interpolation theory, *Proc. Roy. Soc. Edinburgh* **35**, 181–194.

WUNSCH, G. (1963), Über einige Verallgemeinerungen des Abtasttheorems und seine Anwendungen in der Systemtheorie, *Nachrichtentechnik* **13**, 380–382.

YAO, K., AND THOMAS, J. B. (1966), On truncation error bounds for sampling representations of band-limited signals, *IEEE Trans. Aerosp. Electron. Syst.* **AES-2**, 640–647.